$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2.$

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December 12, 2022

Abstract

For a given s and k, we consider the group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. For a subset $\{p, q\} \subset G$, we define the s-fold signed sumset to be

$$[0, s]_{\pm}\{p, q\} = \{\lambda_1 \cdot p + \lambda_2 \cdot q : |\lambda_1| + |\lambda_2| \leq s\}.$$

We ask, for a given k and the G it defines, whether there exist elements $p, q \in G$ such that $[0, s]_{\pm}\{p, q\} = G$. If there does exist such a pair, we write that

$$\phi_{\pm}(G, [0, s]) = 2.$$

We seek all solutions s, k to the above equation. The behavior of the function changes based on the equivalence class mod 4 of s. We place a sharp upper bound on solutions to the equation, prove complete solutions for when s is odd, and prove what we conjecture to be complete solutions for even s.

1 Introduction

Our work focuses on $[0, s]$-fold signed sumsets, and particularly the case where such a sumset contains its entire ambient group. We begin with some definitions.

Definition 1.1. *For a positive m and a nonnegative h, a* layer *of the m-dimensional integer lattice is defined as*

$$\mathbb{Z}^m(h) = \{(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{Z}^m : |\lambda_1| + |\lambda_2| + \ldots + |\lambda_m| = h\}.$$

For a given $s \geq 0$, we also employ an interval notation to describe subsets of the integer lattice

$$\mathbb{Z}^m([0, s]) = \{(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{Z}^m : |\lambda_1| + |\lambda_2| + \ldots + |\lambda_m| \in [0, s]\}.$$

Definition 1.2. *Let s be a positive integer and let $A = \{a_1, a_2, \ldots, a_m\}$. The $[0, s]$-fold signed sumset of A is defined as*

$$[0, s]_{\pm}A = \{\lambda_1 \cdot a_1 + \lambda_2 \cdot a_2 + \cdots + \lambda_m \cdot a_m : (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{Z}^m([0, s])\}.$$

Definition 1.3. *Let s be a positive integer, G be a group, and A a subset of G. Then A* spans *G if and only if $[0, s]_{\pm}A = G$. In this case we call A a* ***spanning set*** *of G, and denote by $\phi_{\pm}$ the size of the smallest spanning set of G for a given s*

$$\phi_{\pm}(G, [0, s]) = \min\{|A| : [0, s]_{\pm}A = G\}.$$

Our work focuses on groups of the form $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ for which $\phi_{\pm}(G, [0, s]) = 2$. We include here Park's results in [2]

Theorem 1.4 (Park, 2020)**.** *Given a positive integer s, let $k = \frac{s^2}{2}$ when s is even and $k = \frac{s^2-1}{2}$ when s is odd. Then $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$, where the spanning set of $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ is $\{(0, 1), (1, s-1)\}$ when s is even and $\{(1, \frac{s-1}{2}), (1, \frac{s+1}{2})\}$ when s is odd.*

Conjecture 1.5 (Park, 2020)**.** *The value of k found in the theorem above is the largest possible k for which $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$.*

2 Main results

Theorem 2.1. Conjecture 1.5 *holds: for any given s the value $k = \left\lfloor \frac{s^2}{2} \right\rfloor$ is the largest k such that*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2.$$

Theorem 2.2. *The equation* $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0,s]) = 2$ *holds for*

- s *is odd and* $k \in \left\{k \in \mathbb{N} : k \leq \left\lfloor \frac{s^2}{2} \right\rfloor \right\}$
- $s \equiv 0 \bmod 4$ *and* $k \in \{k \in \mathbb{N} : k \leq \frac{s^2-s}{2}\} \cup \{k \in \mathbb{N} : k \in \left[\frac{s^2-s}{2}+1, \frac{s^2}{2}\right]$ *and* k *is even*$\}$
- $s \equiv 2 \bmod 4$ *and* $k \in \{k \in \mathbb{N} : k \leq \frac{s^2-s}{2}\} \cup \{k \in \mathbb{N} : k \in \left[\frac{s^2-s}{2}+1, \frac{s^2}{2}\right]$ *and* $k \equiv 2 \bmod 4\}$.

Conjecture 2.3. *The values* s, k *given in* Theorem 2.2 *are the only solutions to the equation* $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0,s]) = 2$.

3 Methods

To prove Theorem 2.1, we first prove some results on the integer lattice $\mathbb{Z}^2([0,s])$. Given a nonnegative integer s, we define two functions $E(s)$ and $O(s)$. $E(s)$ is the number of coefficient pairs in $\mathbb{Z}^2([0,s])$ whose sum is even

$$E(s) = |\{(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s]) \mid |\lambda_1| + |\lambda_2| \equiv 0 \bmod 2\}|$$

while $O(s)$ is the number of coefficient pairs in $\mathbb{Z}^2([0,s])$ whose sum is odd

$$O(s) = |\{(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s]) \mid |\lambda_1| + |\lambda_2| \equiv 1 \bmod 2\}|.$$

For convenience, we call elements of the integer lattice even if the sum $|\lambda_1| + |\lambda_2|$ is even, and call them odd if the sum is odd.

We now prove a lemma concerning these two functions, which will be useful when the parity of $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$ determines some property of a group element corresponding to (λ_1, λ_2).

Lemma 3.1. *The functions* $E(s)$ *and* $O(s)$ *adhere to the following formulae:*

$$E(s) = \begin{cases} s^2 + 2s + 1, & s \equiv 0 \bmod 2 \\ s^2, & s \equiv 1 \bmod 2 \end{cases}$$

$$O(s) = \begin{cases} s^2, & s \equiv 0 \bmod 2 \\ s^2 + 2s + 1, & s \equiv 1 \bmod 2. \end{cases}$$

Proof. We begin with two identities derived from the table found in [1, p. 28] — one concerning the subset $\mathbb{Z}^2([0,s])$ of the integer lattice,

$$|\mathbb{Z}^2([0,s])| = 2s^2 + 2s + 1, \tag{1}$$

and a second concerning the size of an individual layer $\mathbb{Z}^2(h)$ for some $h \geq 0$,

$$|\mathbb{Z}^2(h)| = \begin{cases} 4h, & h \geq 1 \\ 1, & h = 0. \end{cases} \tag{2}$$

Because the set $\mathbb{Z}^2([0,s])$ can be partitioned into even and odd elements, the equation below follows from Equation 1

$$E(s) + O(s) = 2s^2 + 2s + 1. \tag{3}$$

Given any $h \in [0,s]$, it is clear that all the elements of the layer $\mathbb{Z}^2(h)$ will be even if h is even and odd if h is odd. With this fact and Equation 2, we calculate $E(s)$ for even values of s:

$$\begin{aligned} E(s) &= |\mathbb{Z}^2(0)| + |\mathbb{Z}^2(2)| + \cdots + |\mathbb{Z}^2(s)| \\ &= 1 + 4 \cdot 2 + \cdots + 4 \cdot s \\ &= 1 + 4 \cdot (2 + 4 + \cdots + s) \\ &= 1 + 8 \cdot (1 + 2 + \cdots + \frac{s}{2}) \\ &= 1 + 8 \cdot \frac{\frac{s}{2} \cdot (\frac{s}{2} + 1)}{2} \\ &= 1 + 8 \cdot \frac{s^2 + 2s}{8} \\ E(s) &= s^2 + 2s + 1. \end{aligned}$$

By Equation 3, this implies that $O(s) = s^2$ for even values of s.

We now derive the formula for $E(s)$ when s is odd. Clearly no element of the layer $\mathbb{Z}^2(s)$ will be even, so we have:

$$\begin{aligned} E(s) &= E(s-1) \\ E(s) &= (s-1)^2 + 2(s-1) + 1 \\ E(s) &= s^2. \end{aligned}$$

By Equation 3, we conclude that $O(s) = s^2 + 2s + 1$ for odd values of s. □

We now use Lemma 3.1 to prove a fact about spanning pairs of groups $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ with $k > \frac{s^2}{2}$.

Proposition 3.2. *Let k be a positive integer such that $k > \frac{s^2}{2}$, and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. Let $A = \{(1, x), (1, y)\}$ be a subset of G. Then $[0, s]_{\pm}A \neq G$, i.e. A does not span G.*

Proof. Take any $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$, and consider the spanned element

$$(a, b) = \lambda_1 \cdot (1, x) + \lambda_2 \cdot (1, y).$$

Note that the parity of (λ_1, λ_2) corresponds to the value, and therefore the parity, of a — if $|\lambda_1| + |\lambda_2|$ is even, then $a = 0$; if it is odd, then $a = 1$.

We now show that there are either insufficient even elements of $\mathbb{Z}^2([0, s])$ to span the even $(a = 0)$ elements of G, or insufficient odd elements of $\mathbb{Z}^2([0, s])$ to span the odd $(a = 1)$ elements of G.

Recalling that $k > \frac{s^2}{2}$, calculating the size of the group yields

$$|G| = |\mathbb{Z}_2 \times \mathbb{Z}_{2k}| > 2s^2.$$

We partition G by the value of a for each element, which divides the group into two halves, each with more than s^2 elements.

By Lemma 3.1, if s is even, there are $O(s) = s^2$ odd elements of $\mathbb{Z}^2([0, s])$. Because only odd elements (λ_1, λ_2) can span odd elements of G, this implies that A can span at most s^2 odd elements of G, which is insufficient to span G.

Again by Lemma 3.1, if s is odd, there are $E(s) = s^2$ even elements of $\mathbb{Z}^2([0, s])$. In this case, there are insufficient even elements of $\mathbb{Z}^2([0, s])$ to span the even elements of G. Therefore, for any value of s and any $k > \frac{s^2}{2}$, the subset $A = \{(1, x), (1, y)\}$ cannot span G. □

Next, we impose a further restriction on spanning pairs of G.

Proposition 3.3. *Given positive s, k, and a group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$, let $A = \{(0,x),\ (1,y)\}$ be a subset of G. If x is even, then $[0,s]_{\pm}A \neq G$.*

Proof. We prove that if x is even, then $(0,1) \notin [0,s]_{\pm}A$. Suppose indirectly that x is even and that $(0,1) \in [0,s]_{\pm}A$, i.e. there exist some λ_1, λ_2 such that

$$\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = (0,1)$$

with $|\lambda_1| + |\lambda_2| \in [0,s]$. Because the first component of $(0,1)$ is 0, the coefficient λ_2 must be even. The equation determining the second component of the sum is

$$\lambda_1 \cdot x + \lambda_2 \cdot y \equiv 1 \bmod 2k.$$

We have established that λ_2 is even, so if x is also even, then the sum on the left side of the equation must be even, while the right side must be odd, which is impossible. Therefore $(0,1)$ cannot be in the span of A, and $[0,s]_{\pm}A \neq G$. □

A final restriction on potential spanning pairs will put the final proof of the conjecture within reach.

Proposition 3.4. *Let k be a positive integer such that $k > \frac{s^2}{2}$, and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. Let $A = \{(0,x),(1,y)\}$ be a subset of G. If y is odd, then $[0,s]_{\pm}A \neq G$.*

Proof. If x is even, then A does not span G by Proposition 3.3, so we assume that both x and y are odd.

Take any $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$, and consider the spanned element

$$(a,b) = \lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y).$$

Because x and y are both odd, b is even if (λ_1, λ_2) is even, and odd if (λ_1, λ_2) is odd — the parity of the coefficients corresponds exactly with the parity of b.

We now show that there are either insufficient even elements of $\mathbb{Z}^2([0,s])$ to span the elements of G with an even second component b, or insufficient odd elements of $\mathbb{Z}^2([0,s])$ to span the elements of G whose second component is odd.

Recalling that $k > \frac{s^2}{2}$, calculating the size of the group yields

$$|G| = |\mathbb{Z}_2 \times \mathbb{Z}_{2k}| > 2s^2.$$

We partition G by the parity of each element's second component b, which divides the group into two halves, each with more than s^2 elements.

By Lemma 3.1, if s is even, there are $O(s) = s^2$ odd elements of $\mathbb{Z}^2([0, s])$. Because of the established relationship between the parity of (λ_1, λ_2) and the spanned group element, this implies that A can span at most s^2 elements of G whose second component is odd.

Again by Lemma 3.1, if s is odd, there are $E(s) = s^2$ even elements of $\mathbb{Z}^2([0, s])$. In this case, there are insufficient even elements of $\mathbb{Z}^2([0, s])$ to span the elements of G whose second component is even. Therefore, for any value of s and any $k > \frac{s^2}{2}$, the subset $A = \{(0, x), (1, y)\}$ cannot span G if y is odd. □

Proposition 3.5. *Given some positive integers s and k, suppose that $A = \{(0, x), (1, y)\}$ is an s-spanning set for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$, where x is odd and y is even. Then the set $A' = \{(1, x), (1, y)\}$ is also an s-spanning set for G.*

Proof. Because A spans G, there exists some function $f : G \to \mathbb{Z}^2([0, s])$ that, given some $(a, b) \in G$, returns $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ such that

$$\lambda_1 \cdot (0, x) + \lambda_2 \cdot (1, y) = (a, b).$$

We use f to construct an analagous function $g : G \to \mathbb{Z}^2([0, s])$ that, for a given $(a, b) \in G$, returns $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ such that

$$\lambda_1 \cdot (1, x) + \lambda_2 \cdot (1, y) = (a, b),$$

proving that the set $A' = \{(1, x), (1, y)\}$ also spans G.

We begin by defining $g(a, b) = f(a, b)$ for even values of b.

Take some $(a, b) \in G$ and let $(\lambda_1, \lambda_2) = f(a, b)$, i.e.

$$\lambda_1 \cdot (0, x) + \lambda_2 \cdot (1, y) = (a, b).$$

If b is even, then because x is odd and y is even, λ_1 must be even. Consequently, we know that $\lambda_1 \cdot (1, x) = \lambda_1 \cdot (0, x)$, and therefore that

$$\lambda_1 \cdot (1, x) + \lambda_2 \cdot (1, y) = \lambda_1 \cdot (0, x) + \lambda_2 \cdot (1, y)$$
$$\lambda_1 \cdot (1, x) + \lambda_2 \cdot (1, y) = (a, b),$$

so $g(a,b) = f(a,b)$ for even values of b.

Take some $(a,b) \in G$ where b is odd and let $(\lambda_1, \lambda_2) = f(a,b)$, i.e.

$$\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = (a,b).$$

When b is odd, then because x is odd and y is even, λ_1 must also be odd. In this case, we define $g(a,b)$ as

$$g(0,b) = f(1,b) \quad \text{and} \quad g(1,b) = f(0,b).$$

We begin by proving that when $a = 0$, the function g satisfies the desired properties. Let $(\lambda_1, \lambda_2) = f(1,b)$ for an odd b. By the definition of f

$$\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = (1,b),$$

so λ_2 must be odd. Because λ_1 and λ_2 are both odd, the sum $\lambda_1 + \lambda_2$ must be even. Therefore

$$\lambda_1 \cdot (1,x) + \lambda_2 \cdot (1,y) = (0,b),$$

and we define $g(0,b) = f(1,b)$ when b is odd.

We now prove that $g(1,b) = f(0,b)$ for odd b. Let $(\lambda_1, \lambda_2) = f(0,b)$ for some odd b. By the definition of f

$$\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = (0,b),$$

so λ_2 must be even. Because λ_1 is odd and λ_2 is even, the sum $\lambda_1 + \lambda_2$ must be odd. Therefore

$$\lambda_1 \cdot (1,x) + \lambda_2 \cdot (1,y) = (1,b),$$

and we define $g(1,b) = f(0,b)$ when b is odd.

We have now proved that the function $g : G \to \mathbb{Z}^2([0,s])$ defined by the formula

$$g(a,b) = \begin{cases} f(a,b), & b \text{ is even} \\ f(1,b), & b \text{ is odd, } a = 0 \\ f(0,b), & b \text{ is odd, } a = 1 \end{cases}$$

satisfies the desired properties, meaning that the set $A' = \{(1,x),(1,y)\}$ spans G. $\square$

We are now ready to prove Theorem 2.1.

Theorem 2.1. Conjecture 1.5 *holds: for any given s the value $k = \left\lfloor \frac{s^2}{2} \right\rfloor$ is the largest k such that*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2.$$

Proof. Let s be a positive integer, let $k > \frac{s^2}{2}$, and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. Clearly no subset of the form $\{(0, x), (0, y)\}$ can span G, and by Propositions 3.2, 3.3, and 3.4, we know that for G, any spanning pair must have the form $A = \{(0, x), (1, y)\}$ for some odd x and even y.

If such a spanning pair existed, however, that would imply by Proposition 3.5 that the set $A' = \{(1, x), (1, y)\}$ also spans G. Because Proposition 3.2 proved this impossible, we have shown that A cannot span G, and therefore no subset of two elements can span G. □

Having proven Theorem 2.1, we now prove all of the solutions to the equation found in Theorem 2.2.

Proposition 3.6. *Let s be a positive integer, and let d, x, y be positive integers such that*

- *$s^2 - d^2$ is even*
- *x is odd*
- *$x + y = s$*
- *x and y are coprime*
- *$4xy = s^2 - d^2$*

then the group $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$ is s-spanned by the pair of elements $\{(0, x), (1, y)\}$; therefore

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}, [0, s]) = 2.$$

Proof. For an arbitrary element $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$, we first show that there are coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $\lambda_1 \cdot (0, x) + \lambda_2 \cdot (1, y) = (a, b)$.

The span of $(0,x)$ will form a subgroup $H \leq G$ of order $\frac{s^2-d^2}{x} = 4y$. This subgroup has $\frac{|G|}{4y} = 2x$ corresponding cosets. The element (a,b) that we wish to span must lie in one of these cosets, so we first show that each of the cosets can be reached by some multiple $\lambda_2 \cdot (1,y)$.

For each $\mu \in [0, 2x-1]$, the multiple $\mu \cdot (1,y)$ reaches a different coset of H, implying that this set of multiples reaches all $2x$ cosets of H: take two distinct $\mu_1, \mu_2 \in [0, 2x-1]$ and assume without loss of generality that $\mu_1 > \mu_2$. $\mu_1 \cdot (1,y)$ and $\mu_2 \cdot (1,y)$ belong to different cosets because $\mu_1 \cdot (1,y) - \mu_2 \cdot (1,y) = (\mu_1 - \mu_2) \cdot (1,y) \notin H$. To see this, let $\mu' = \mu_1 - \mu_2 \in [1, 2x-1]$ and suppose for contradiction that $\mu' \cdot (1,y) \in H$. This would imply that

$$\mu' \cdot (1,y) = c \cdot (0,x)$$

for some integer c. Because x and y are coprime, the only $\mu' \in [1, 2x-1]$ that could satisfy the above equation is x. But because x is odd, we know that

$$x \cdot (1,y) = (1, xy) \neq c \cdot (0,x)$$

for any c. We therefore conclude that $\mu \cdot (1,y)$ spans a different coset of H for each $\mu \in [0, 2x-1]$, and consequently that they span every coset.

We return to our element $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$. It must lie in some coset of H, so by our findings above there must be some $\lambda_2 \in [0, 2x-1]$ such that $\lambda_2 \cdot (1,y)$ is in this same coset. Because each of these cosets is of size $4y$, there must be some $\lambda_1 \in [-2y+1, 2y]$ such that

$$\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = (a,b).$$

There is no guarantee, however, that $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$. Based on the constraints above, we have only that

$$|\lambda_1| + |\lambda_2| \leq 2y + 2x - 1 = 2s - 1.$$

If $|\lambda_1| + |\lambda_2| \leq s$, then we have found coefficients in $\mathbb{Z}^2([0,s])$ that span (a,b) and are done.

If, however, $|\lambda_1| + |\lambda_2| \in [s+1, 2s-1]$, we show that there exist $(\lambda_1', \lambda_2') \in \mathbb{Z}^2([0,s])$ that span the same element (a,b). We select these values as follows:

$$\lambda_1' = \begin{cases} \lambda_1 - 2y, & \lambda_1 \geq 0 \\ \lambda_1 + 2y, & \lambda_1 < 0 \end{cases} \qquad \lambda_2' = \lambda_2 - 2x.$$

This selection of variables implies that $|\lambda_1'| = 2y - |\lambda_1|$ and $|\lambda_2'| = 2x - |\lambda_2|$. Therefore

$$\begin{aligned}
|\lambda_1'| + |\lambda_2'| &= 2y - |\lambda_1| + 2x - |\lambda_2| \\
|\lambda_1'| + |\lambda_2'| &= 2(x+y) - (|\lambda_1| + |\lambda_2|) \\
|\lambda_1'| + |\lambda_2'| &= 2s - (|\lambda_1| + |\lambda_2|)\,.
\end{aligned}$$

Because $|\lambda_1| + |\lambda_2| \in [s+1, 2s-1]$, this implies that

$$|\lambda_1'| + |\lambda_2'| \in [1, s-1],$$

placing (λ_1', λ_2') within the acceptable bounds for $\mathbb{Z}^2([0,s])$.

It remains only to prove that (λ_1', λ_2') span the same element (a,b) as the original coefficients. If $\lambda_1 \geq 0$, meaning $\lambda_1' = \lambda_1 - 2y$, then

$$\begin{aligned}
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (\lambda_1 - 2y) \cdot (0,x) + (\lambda_2 - 2x) \cdot (1,y) \\
&= [\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y)] - [2y \cdot (0,x) + 2x \cdot (1,y)] \\
&= (a,b) - (0, 4xy) \\
&= (a,b) - (0, s^2 - d^2) \\
&= (a,b) - (0,0) \\
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (a,b).
\end{aligned}$$

If $\lambda_1 < 0$, meaning $\lambda_1' = \lambda_1 + 2y$, then

$$\begin{aligned}
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (\lambda_1 + 2y) \cdot (0,x) + (\lambda_2 - 2x) \cdot (1,y) \\
&= [\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y)] - 2y \cdot (0,x) + 2x \cdot (1,y) \\
&= (a,b) - (0, 2xy) + (0, 2xy) \\
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (a,b).
\end{aligned}$$

Since in either case, the new $(\lambda_1', \lambda_2') \in \mathbb{Z}^2([0,s])$ spans the same element (a,b), we have that our arbitrary element $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$ is s-spanned by the elements $(0,x)$ and $(1,y)$, as was to be shown. □

Lemma 3.7. *Let s be a positive integer, $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ be a group, and $A = \{p, q\}$ be a pair of elements. Then A is a $[0,s]$ signed spanning set for G if and only if it spans the subset $\mathbb{Z}_2 \times \{0, 1, \ldots, k\} \subset G$.*

Proof. The "only if" direction is clearly true, so we prove the "if" statement. For any $g \in G$, either g or $-g$ is in the set $\mathbb{Z}_2 \times \{0, 1, \ldots, k\}$ In the latter case, take the coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ that span $-g$ and observe that

$$\begin{aligned} -\lambda_1 \cdot p + -\lambda_2 \cdot q &= -(\lambda_1 \cdot p + \lambda_2 \cdot q) \\ -\lambda_1 \cdot p + -\lambda_2 \cdot q &= -(-g) \\ -\lambda_1 \cdot p + -\lambda_2 \cdot q &= g. \end{aligned}$$

Therefore g can also be spanned by the spanning set A, proving our claim. □

Definition 3.8. *Given some $s \geq 1$, a group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$, a pair of elements $A = \{(a, x), (b, y)\}$, and some $(c, z) \in G$, we say that the element (c, z) is **directly spanned** if there exist $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ such that*

$$\lambda_1 \cdot a + \lambda_2 \cdot b \equiv c \bmod 2 \quad \textit{and } \lambda_1 \cdot x + \lambda_2 \cdot y = z.$$

Definition 3.9. *Given some $s \geq 1$, a group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$, a pair of elements $A = \{(a, x), (b, y)\}$, and some $(c, z) \in G$, we say that the element (c, z) is **negatively spanned** if there exist $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ such that*

$$\lambda_1 \cdot a + \lambda_2 \cdot b \equiv c \bmod 2 \quad \textit{and } \lambda_1 \cdot x + \lambda_2 \cdot y = -1 \cdot 2k + z.$$

Lemma 3.10. *Let $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ be a group with a pair of elements $A = \{(a, x), (b, y)\}$ that directly s-span some element $(c, z) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. Then for any $k' \in \mathbb{N}$ the corresponding element $(c, z) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k'}$ is also directly s-spanned by A.*

Proof. The first component of the element spanned $\lambda_1 \cdot a + \lambda_2 \cdot b \equiv c \bmod 2$ will clearly not change between the two groups, and by the definition of direct spanning we have that

$$\lambda_1 \cdot x + \lambda_2 \cdot y = z \equiv z \bmod 2k',$$

so the element (c, z) is directly spanned in both groups. □

Proposition 3.11. *Given an odd s, the equation*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$$

holds if and only if $k \in [1, \frac{s^2-1}{2}]$.

Proof. By Theorem 2.1, the equation does not hold for any $k > \frac{s^2-1}{2}$. It now remains to prove the "if" direction.

Given some odd s, we let $k = \frac{s^2-1}{2}$ and let

$$x = \begin{cases} \frac{s+1}{2}, & s \equiv 1 \bmod 4 \\ \frac{s-1}{2}, & s \equiv 3 \bmod 4 \end{cases} \qquad y = \begin{cases} \frac{s-1}{2}, & s \equiv 1 \bmod 4 \\ \frac{s+1}{2}, & s \equiv 3 \bmod 4. \end{cases}$$

Our choice of x and y satisfies the hypothesis of Proposition 3.6, which we apply to prove that the set $A = \{(0, x), (1, y)\}$ s-spans $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-1}$. Now we prove that the subset $\mathbb{Z}_2 \times \{0, 1, \ldots, \frac{s^2-1}{2} - 1\}$ is directly spanned by A, which by Lemma 3.7 and Lemma 3.10 suffices to prove our claim.

For a given $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ that spans a certain element of $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-1}$, we let μ_1 be the coefficient corresponding to $\frac{s-1}{2}$ and μ_2 be the one corresponding to $\frac{s+1}{2}$, i.e.

$$\mu_1 = \begin{cases} \lambda_2, & s \equiv 1 \bmod 4 \\ \lambda_1, & s \equiv 3 \bmod 4 \end{cases} \qquad \mu_2 = \begin{cases} \lambda_1, & s \equiv 1 \bmod 4 \\ \lambda_2, & s \equiv 3 \bmod 4. \end{cases}$$

Because $\mu_1 \cdot \frac{s-1}{2} + \mu_2 \cdot \frac{s+1}{2} \geq \frac{-s^2-s}{2}$ for all $(\mu_1, \mu_2) \in \mathbb{Z}^2([0, s])$, then for any $(a, b) \in \mathbb{Z}_2 \times \{0, 1, \ldots, \frac{s^2-1}{2} - 1\}$ that is negatively spanned by such a (μ_1, μ_2), we have that $b \in [\frac{s^2-s-2}{2}, \frac{s^2-1}{2} - 1]$. For a negatively spanned b in this range we know that $\mu_1 + \mu_2 = -s$. For suppose that $\mu_1 + \mu_2 \geq -s + 1$, and observe that

$$\mu_1 \cdot \frac{s-1}{2} + \mu_2 \cdot \frac{s+1}{2} \geq (-s+1) \cdot \frac{s+1}{2} = \frac{-s^2+1}{2} \equiv \frac{s^2-1}{2} \bmod s^2 - 1,$$

which is outside of our established range for negatively spanned b.

For a given (a, b) negatively spanned by some $(\mu_1, \mu_2) \in \mathbb{Z}^2([0, s])$, we divide the remaining work into two cases. In the case where $\mu_2 = -s$ and therefore $\mu_1 = 0$, we have that

$$(s-2) \cdot \frac{s+1}{2} = \frac{s^2-s-2}{2}$$

which is equivalent $\bmod s^2 - 1$ to

$$-s \cdot \frac{s+1}{2} = \frac{-s^2-s}{2} \equiv \frac{s^2-s-2}{2} \bmod s^2 - 1.$$

Furthermore, because $-s \equiv (s-2) \bmod 2$ the coefficients $\mu_1' = 0, \mu_2' = s - 2$ will directly span the same element of $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-1}$ which the coefficients $\mu_1 = 0, \mu_2 = -s$ negatively span.

In the second case, where $\mu_2 \geq -s+1$ and therefore $\mu_1 \leq -1$, let $\mu_1' = \mu_1 + s + 1$ and $\mu_2' = \mu_2 + s - 1$. We first note that

$$\begin{aligned}
\mu_1' \cdot \frac{s-1}{2} + \mu_2' \cdot \frac{s+1}{2} &= (\mu_1 + s + 1) \cdot \frac{s-1}{2} + (\mu_2 + s - 1) \cdot \frac{s+1}{2} \\
&= \left(\mu_1 \cdot \frac{s-1}{2} + \mu_2 \cdot \frac{s+1}{2}\right) + \frac{(s+1)(s-1)}{2} + \frac{(s-1)(s+1)}{2} \\
&= b - (s^2 - 1) + (s^2 - 1) \\
&= b.
\end{aligned}$$

Taken together with the fact that $\mu_1' \equiv \mu_1 \bmod 2$ and $\mu_2' \equiv \mu_2 \bmod 2$, the above implies that (μ_1', μ_2') directly spans the element (a, b) in question. We now prove that $(\mu_1', \mu_2') \in \mathbb{Z}^2([0, s])$, keeping in mind that $\mu_2 \geq -s + 1$

$$\begin{aligned}
|\mu_1'| + |\mu_2'| &= |\mu_1 + s + 1| + |\mu_2 + s - 1| \\
&= (\mu_1 + s + 1) + (\mu_2 + s - 1) \\
&= (\mu_1 + \mu_2) + s + 1 + s - 1 \\
&= -s + 2s - 2 \\
&= s - 2.
\end{aligned}$$

We have shown that our new coefficients $(\mu_1', \mu_2') \in \mathbb{Z}^2([0, s])$ directly span the element in question (a, b). This proves that any element in our subset is directly spanned by A, which as shown above suffices to prove our claim. □

Proposition 3.12. *For a given positive integer $s \equiv 0 \bmod 4$, let $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$. Then the pair of elements $A = \{(0, x), (1, y)\}$ where*

$$x = \frac{s-2}{2} \quad \text{and} \quad y = \frac{s+2}{2}$$

is an s-spanning pair for G. Therefore

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}, [0, s]) = 2.$$

Proof. Our proposition is a particular case of Proposition 3.6, where $d = 2$. Because $s \equiv 0 \bmod 4$, we know that x and y are both odd; because they differ by 2, this further implies that they are coprime. The hypothesis of Proposition 3.6 thus holds, proving our claim. □

Proposition 3.13. *For a given positive integer $s \equiv 0 \bmod 4$, take any $k \leq \frac{s^2-s}{2}$. The group $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ can be s-spanned by the pair $A = \{(0, x), (1, y)\}$ where*

$$x = \frac{s-2}{2} \quad \text{and} \quad y = \frac{s+2}{2},$$

and consequently $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$.

Proof. It suffices by Lemma 3.7 and Lemma 3.10 to show that for $k = \frac{s^2-s}{2}$, the subset $\mathbb{Z}_2 \times \{0, 1, \ldots, k\}$ can be *directly spanned* by A. We proved above in Proposition 3.12 that the group $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ is spanned by A. Observe that for any $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$

$$\frac{-s^2 - 2s}{2} \leq \lambda_1 \cdot x + \lambda_2 \cdot y \leq \frac{s^2 + 2s}{2},$$

so any element $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ that is not directly spanned will instead be negatively spanned by Definition 3.9 above. Taking an arbitrary $(a, b) \in \mathbb{Z}_2 \times \{0, \ldots, \frac{s^2-s}{2}\}$ that is negatively spanned by some $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$, we will show that this same element is directly spanned by the coefficients $(\lambda_1, \lambda_2 + 4x) \in \mathbb{Z}^2([0, s])$. We first observe that

$$\begin{aligned}
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= (\lambda_1 \cdot x + \lambda_2 \cdot y) + 4x \cdot y \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= -1 \cdot (s^2 - 4) + b + s^2 - 4 \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= b.
\end{aligned}$$

Because $4x$ is even, we also have that $\lambda_2 + 4x \equiv a \bmod 2$, meaning the new coefficients directly span (a, b). It still remains to be shown that the new coefficients are in $\mathbb{Z}^2([0, s])$.

Because the element is negatively spanned and $b \in [0, \frac{s^2-s}{2}]$, we know that its coefficients were generated by the second step in Proposition 3.6, so $\lambda_2 < 0$ and $|\lambda_1| + |\lambda_2| \in [1, s-1]$. We will prove that $\lambda_2 \leq -2x$, which implies that the above coefficients are also in the bounds i.e.

$$|\lambda_1| + |\lambda_2 + 4x| \leq s.$$

Suppose that $\lambda_2 = -2x + 1 = -s + 3$ and that $\lambda_1 = -2$. This is the coefficient pair with the lowest spanned value $\lambda_1 \cdot x + \lambda_2 \cdot y$ such that $-2x < \lambda_2 < 0$ and $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([1, s-1])$. Calculating this value

$$\begin{aligned}
\lambda_1 \cdot x + \lambda_2 \cdot y &= -2 \cdot \frac{s-2}{2} + (-s+3) \cdot \frac{s+2}{2} \\
&= -s + 2 + \frac{-s^2 + s + 6}{2} \\
&= \frac{-s^2 - s + 10}{2} \\
&\equiv \frac{s^2 - s + 2}{2} \bmod s^2 - 4,
\end{aligned}$$

we see that it is outside the assumed range $b \in [0, \frac{s^2-s}{2}]$. Therefore for all $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([1, s-1])$ that negative span an element with b in this range, we know that $\lambda_2 \leq -2x$. Hence $(\lambda_1, \lambda_2 + 4x) \in \mathbb{Z}^2([0, s])$ directly spans the element (a, b) while staying within the bounds for spanning coefficients, which as shown above suffices to prove our claim. □

Lemma 3.14. *Let $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ such that $b \in [\frac{s^2-s+2}{2}, \frac{s^2+s-4}{2}]$. If (a, b) is negatively spanned by the pair A that spans $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$, then A also negatively spans $(a, b+4)$.*

Proof. Take any negatively spanned element (a, b) within the specified range, and consider its spanning coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$. We first observe that the coefficients $(\lambda_1 - 2, \lambda_2 + 2)$ will span the element $(a, b+4)$, as $\lambda_2 + 2 \equiv \lambda_2 \equiv a \bmod 2$, and

$$(\lambda_1 - 2) \cdot \frac{s-2}{2} + (\lambda_1 + 2) \cdot \frac{s+2}{2} = b - 2 \cdot \frac{s-2}{2} + 2 \cdot \frac{s+2}{2}$$
$$(\lambda_1 - 2) \cdot \frac{s-2}{2} + (\lambda_1 + 2) \cdot \frac{s+2}{2} = b + 4.$$

We now prove that these coefficients are also in $\mathbb{Z}^2([0, s])$. We begin by proving that $\lambda_2 \leq -2$. First, if $\lambda_2 = 0$, then any value of λ_1 can not span (a, b), for

$$\begin{aligned}\lambda_1 \cdot x + 0 \cdot y &\geq -s \cdot \frac{s-2}{2} \\ &= \frac{-s^2 + 2s}{2} \\ &\equiv \frac{s^2 + 2s - 8}{2} \bmod s^2 - 4\end{aligned}$$

which is outside of the specified range for b. Second, if $\lambda_2 \neq 0$ but $\lambda_2 \geq -1$, then $\lambda_1 \geq -s + 1$ which implies

$$\begin{aligned}\lambda_1 \cdot x + \lambda_2 \cdot y &\geq (-s+1) \cdot \frac{s-2}{2} + -1 \cdot \frac{s+2}{2} \\ &= \frac{-s^2 + 3s - 2}{2} - \frac{s+2}{2} \\ &= \frac{-s^2 + 2s - 4}{2} \\ &\equiv \frac{s^2 + 2s - 12}{2} \bmod s^2 - 4,\end{aligned}$$

which is also outside of the specified range for b. We have proved that $\lambda_2 \le -2$, implying that $|\lambda_2 + 2| = |\lambda_2| - 2$. Clearly we also have that $|\lambda_1 - 2| \le |\lambda_1| + 2$, meaning

$$|\lambda_1 - 2| + |\lambda_2 + 2| \le |\lambda_1| + 2 + |\lambda_2| - 2 = |\lambda_1| + |\lambda_2| \le s.$$

Therefore $(\lambda_1 - 2, \lambda_2 + 2) \in \mathbb{Z}^2([0, s])$, proving our claim. □

Proposition 3.15. *For a positive integer* $s \equiv 0 \bmod 4$*, let* k *be an even integer* $k \le \frac{s^2-4}{2}$*. Then the pair* A *from above* s*-spans* $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ *implying*

$$\phi_\pm(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2.$$

Proof. Take some $i \in \mathbb{N}$, and let $k_i = \frac{s^2-4-4i}{2}$. We know by Proposition 3.13 that all elements (a, b) with $b \le \frac{s^2-s}{2}$ can be directly spanned by A, and thus are spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ for any value of k. Next, we prove that A spans any $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$ such that $b \in [\frac{s^2-s+2}{2}, k_i]$. We first note that if such an element exists, then $4i < s - 4$. If this element is directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$, then it is also directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. If it is negatively spanned, then a more involved argument is required.

We prove that the coefficients that span $(a, b + 4i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ will span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. First, consider the coefficients (λ_1, λ_2) that negatively span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$.

Because $b \in [\frac{s^2-s+2}{2}, k_i]$, we have that

$$\begin{aligned} k_i &\ge \frac{s^2 - s + 2}{2} \\ \frac{s^2 - 4 - 4i}{2} &\ge \frac{s^2 - s + 2}{2} \\ 4i &< s - 4. \end{aligned}$$

We inductively apply Lemma 3.14 up to $(a, b + 4i)$ and call its spanning coefficients (μ_1, μ_2). The lemma holds for all $b + 4,\ b + 8,\ \ldots, b + 4i$ because $4i < s - 4$ implies that $b + 4i < \frac{s^2+s-6}{2}$, within the range where Lemma 3.14 applies.

Finally, we show that the coefficients (μ_1, μ_2) that span $(a, b + 4i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ will span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. Clearly the value a of the spanned element will not change between the two groups, as the spanning pair A and the value μ_2 have not.

Next, because the coefficients negatively span $(a, b+4i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$, we have

$$\mu_1 \cdot \frac{s-2}{2} + \mu_2 \cdot \frac{s+2}{2} = -1 \cdot (s^2-4) + b + 4i$$
$$\mu_1 \cdot \frac{s-2}{2} + \mu_2 \cdot \frac{s+2}{2} = -1 \cdot (s^2-4-4i) + b,$$

meaning $(\mu_1, \mu_2) \in \mathbb{Z}^2([0,s])$ will span the element $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. □

Proposition 3.16. *Let $s \equiv 2 \bmod 4$ be a positive integer, and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$. Then the pair of elements $A = \{(0,x),(1,y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$ spans G, meaning*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}, [0,s]) = 2.$$

Proof. Our proposition is a particular case of Proposition 3.6, where $d = 4$. Because $s \equiv 2 \bmod 4$, we know that x and y are both odd; because they differ by 4, this further implies that they are coprime. The hypothesis of Proposition 3.6 thus holds, proving our claim. □

Proposition 3.17. *Let $s \equiv 2 \bmod 4$ be a positive integer, and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$. Then any $(a,b) \in G$ with $b \in \{0, 1, \ldots, \frac{s^2-4s+6}{2}\}$ can be directly spanned by $A = \{(0,x),(1,y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$.*

Proof. Proposition 3.16 establishes that A spans the group G. Each element in our specified subset is either directly or negatively spanned, so we show that the negatively spanned ones have another set of spanning coefficients in $\mathbb{Z}^2([0,s])$ that directly span them.

Let $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$ be the coefficients that negatively span some element (a,b) with $b \leq \frac{s^2-4s+6}{2}$. We first show that $\lambda_2 \leq -2x$. Assume for contradiction that $\lambda_2 \geq -2x+1 = -s+5$, and therefore that $\lambda_1 \geq -5$. This implies that

$$\lambda_1 \cdot x + \lambda_2 \cdot y \geq (-5) \cdot \frac{s-4}{2} + (-s+5) \cdot \frac{s+4}{2}$$
$$\lambda_1 \cdot x + \lambda_2 \cdot y \geq \frac{-5s+20}{2} - \frac{s^2-s-20}{2}$$
$$\lambda_1 \cdot x + \lambda_2 \cdot y \geq \frac{-s^2-4s+40}{2} \equiv \frac{s^2-4s+8}{2} \bmod s^2-16,$$

which is higher than the assumed value $b \leq \frac{s^2-4s+6}{2}$. Therefore $\lambda_2 \leq -2x$, meaning $|\lambda_1| + |\lambda_2 + 4x| \leq |\lambda_1| + |\lambda_2| \leq s$.

We have established that $(\lambda_1, \lambda_2 + 4x) \in \mathbb{Z}^2([0,s])$, and now show that these new coefficients directly span (a,b). Because λ_2 and $\lambda_2 + 4x$ have the same parity, the first component a of the spanned element remains unchanged. To see that the same b is directly spanned, observe that

$$\begin{aligned}
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= (\lambda_1 \cdot x + \lambda_2 \cdot y) + 4x \cdot y \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= -1 \cdot (s^2 - 16) + b + s^2 - 16 \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= b.
\end{aligned}$$

Therefore the same element (a,b) is also directly spanned by the pair A, as was to be shown. □

Lemma 3.18. *For some $s \equiv 2 \bmod 4$, let $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ and $A = \{(0,x),(1,y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$. For any $(a,b) \in G$ with $b \in [\frac{s^2-4s+8}{2}, \frac{s^2+4s-42}{2}]$ that is negatively spanned by coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$, the coefficients $(\lambda_1 - 2, \lambda_2 + 2)$ negatively span the element $(a, b+8)$ and are also in $\mathbb{Z}^2([0,s])$.*

Proof. We first show that if the coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$ negatively span (a,b), then $\lambda_2 \leq -2$. Supposing for contradiction that it isn't, we split the scenario into two cases: $\lambda_2 = 0$ and $\lambda_2 \geq -1$ but $\lambda_2 \neq 0$.

In the first case, we have that $\lambda_1 \geq -s$, implying

$$\begin{aligned}
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq -s \cdot \frac{s-4}{2} \\
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq \frac{-s^2+4s}{2} \equiv \frac{s^2+4s-32}{2} \bmod s^2 - 16,
\end{aligned}$$

which exceeds our presumed range for b. In the second case where $\lambda_2 \neq 0$, we must have $\lambda_1 \geq -s+1$ and therefore

$$\begin{aligned}
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq -s + 1 \cdot \frac{s-4}{2} + -1 \cdot \frac{s+4}{2} \\
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq \frac{-s^2+5s-4}{2} + \frac{-s-4}{2} \\
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq \frac{-s^2+4s-8}{2} \equiv \frac{s^2+4s-40}{2} \bmod s^2 - 16,
\end{aligned}$$

which also exceeds our presumed range for b. Therefore we must have $\lambda_2 \leq -2$. This bound implies that

$$|\lambda_1 - 2| + |\lambda_2 + 2| \leq |\lambda_1| + 2 + |\lambda_2| - 2 = |\lambda_1| + |\lambda_2| \leq s$$

and thus $(\lambda_1 - 2, \lambda_2 + 2) \in \mathbb{Z}^2([0, s])$. To conclude our argument we show that these coefficients span $(a, b + 8)$. First, $\lambda_2 + 2 \equiv a \bmod 2$ because it has the same parity as λ_2, and thus the first component a is still spanned. For the second component b we have

$$\begin{aligned}(\lambda_1 - 2) \cdot x + (\lambda_2 + 2) \cdot y &= (\lambda_1 \cdot x + \lambda_2 \cdot y) - 2 \cdot x + 2 \cdot y \\ &= b - (s^2 - 16) - (s - 2) + (s + 2) \\ &= b - (s^2 - 16) + 4 \\ &= b + 4 - (s^2 - 16) \equiv b + 4 \bmod s^2 - 16.\end{aligned}$$

□

Proposition 3.19. *Take any $i \geq 0$, and let $k_i = \frac{s^2-16}{2} - 4i$. Then the group $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$ is s-spanned by $A = \{(0, x), (1, y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$.*

Proof. Proposition 3.16 proves the case where $i = 0$. Now consider some $i \geq 1$ and its corresponding k_i and $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. We will prove that the subset $\mathbb{Z}_2 \times \{0, \ldots, k_i\}$ of this group is spanned by A, which suffices by Lemma 3.7 to prove that A spans the entire group. Any elements (a, b) of this subset that can be directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ will also be directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$, so we focus our attention on elements that can only be negatively spanned in our subset of $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$. By Lemma 3.18, this implies that $b \in [\frac{s^2-4s+8}{2}, k_i]$. If such a b exists, it follows that

$$\begin{aligned}\frac{s^2 - 4s + 8}{2} &\leq k_i = \frac{s^2 - 16 - 8i}{2} \\ s^2 - 4s + 8 &\leq s^2 - 16 - 8i \\ 8i &\leq 4s - 24.\end{aligned}$$

We show by induction that $(a, b + 8i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ can also be negatively spanned. We repeatedly apply Lemma 3.18 to (a, b), then $(a, b+8)$, and so on up to $(a, b+8(i-1))$ to prove that $(a, b+8i)$ is negatively spanned. The hypothesis of the lemma holds for all relevant values $b+8, \ldots, b+8(i-1)$ because by our inequality above and the given range for b we have

$$\begin{aligned}b + 8(i - 1) &\leq k_i + 8(i - 1) \\ b + 8(i - 1) &\leq \frac{s^2 - 16 - 8i}{2} + 8(i - 1) \\ b + 8(i - 1) &\leq \frac{s^2 + 8i - 32}{2} \\ b + 8(i - 1) &\leq \frac{s^2 + 4s - 56}{2} < \frac{s^2 + 4s - 40}{2},\end{aligned}$$

placing $b + 8(i-1)$ within the acceptable range.

We have proven that $(a, b+8i)$ can be negatively spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ by some coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$. These coefficients will also span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i} = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16-8i}$, as they will clearly span the same value a for the first component, and for the second component will span

$$\lambda_1 \cdot x + \lambda_2 \cdot y = -1 \cdot (s^2 - 16) + b + 8i$$
$$\lambda_1 \cdot x + \lambda_2 \cdot y = -1 \cdot (s^2 - 16 - 8i) + b,$$

consequently spanning $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. Therefore every element in $\mathbb{Z}_2 \times \{0, \ldots, k_i\}$ is either directly spanned or negatively spanned by our pair A, implying that $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$ is s-spanned by A. □

Proposition 3.20. *Given some* $s \equiv 2 \bmod 4$ *and any* $k \leq \frac{s^2-s}{2}$*, let* $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$*. Then the subset* $A = \{(0, x), (1, y)\}$ *where* $x = \frac{s}{2}$ *and* $y = \frac{s-2}{2}$ *s-spans* G*, meaning*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2.$$

Proof. Because $s-1$ is odd, we apply Proposition 3.11 and find that the set A $(s-1)$-spans the group $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-2s}$. We will prove that A directly s-spans the subset of this group $\mathbb{Z}_2 \times \{0, 1, \ldots, \frac{s^2-s}{2}\}$, which suffices to prove our claim by Lemma 3.7 and Lemma 3.10.

By Proposition 3.11, the subset $\mathbb{Z}_2 \times \{0, 1, \ldots, \frac{s^2-2s}{2}\}$ is already known to be directly spanned. We divide the rest of the elements in our subset of interest into four categories:

1. $(0, \frac{s^2-2s}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s-2}{4}$;
2. $(1, \frac{s^2-2s}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s-2}{4}$;
3. $(0, \frac{s^2-2s-2}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s+2}{4}$; and
4. $(1, \frac{s^2-2s-2}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s+2}{4}$.

We prove that each subset in turn can be directly s-spanned by A.

Case 1

First, note that for a given $i \leq \frac{s-2}{4}$, we have that

$$\begin{aligned} 2i \cdot x + (s-2i) \cdot y &= \frac{2i \cdot s}{2} + \frac{(s-2i)(s-2)}{2} \\ &= \frac{2i \cdot s}{2} + \frac{s^2 - 2s - 2is + 4i}{2} \\ &= \frac{s^2 - 2s}{2} + 2i, \end{aligned}$$

and that $s + 2i \equiv 0 \bmod 2$. Therefore the coefficients $\lambda_1 = 2i,\ \lambda_2 = s - 2i$ span the desired element. To see that they are in $\mathbb{Z}^2([0, s])$, observe that for $i \leq \frac{s-2}{4}$ we have

$$|2i| + |s - 2i| = 2i + s - 2i = s,$$

completing our proof for this case.

Case 2

First note that for a given $i \leq \frac{s-2}{4}$, we have that

$$\begin{aligned} (y + 2i) \cdot x + (x - 2i) \cdot y &= \frac{(s - 2 + 4i) \cdot s}{4} + \frac{(s - 4i) \cdot (s - 2)}{4} \\ &= \frac{s^2 - 2s + 4is}{4} + \frac{s^2 - 2s - 4is + 8i}{4} \\ &= \frac{s^2 - 2s}{2} + 2i, \end{aligned}$$

and that $x \equiv 1 \bmod 2$. Therefore the coefficients $\lambda_1 = y,\ \lambda_2 = x$ span the desired element. To see that they are in $\mathbb{Z}^2([0, s])$, observe that for $i \leq \frac{s-2}{4}$ we have

$$|y + 2i| + |x - 2i| = y + 2i + x - 2i = x + y = s - 1,$$

completing our proof for this case.

Case 3

First, note that for a given $i \leq \frac{s+2}{4}$, we have that

$$\begin{aligned} (y - 1 + 2i) \cdot x + (x + 1 - 2i) \cdot y &= (xy - x + 2ix) + (xy + y - 2iy) \\ &= 2xy + (y - x) + 2i(x - y) \\ &= \frac{s^2 - 2s}{2} - 1 + 2i \end{aligned}$$

and that $x+1-2i \equiv 0 \bmod 2$. Therefore the coefficients $\lambda_1 = y-1+2i$, $\lambda_2 = x+1-2i$ span the desired element. To see that they are in $\mathbb{Z}^2([0,s])$, observe that for $i \leq \frac{s+2}{4}$ we have

$$|y-1+2i| + |x+1-2i| = \frac{s-4+4i}{2} + \frac{s+2-4i}{2} = s-1,$$

completing our proof for this case.

Case 4

First, note that for a given $i \leq \frac{s+2}{4}$, we have that

$$\begin{aligned}(-1+2i)\cdot x + (s+1-2i)\cdot y &= sy + (y-x) + 2i(x-y) \\ &= \frac{s^2-2s}{2} - 1 + 2i,\end{aligned}$$

and that $s+1-2i \equiv 1 \bmod 2$. Therefore the coefficients $\lambda_1 = -1+2i$, $\lambda_2 = s+1-2i$ span the desired element. To see that they are in $\mathbb{Z}^2([0,s])$, observe that for $i \leq \frac{s+2}{4}$ we have

$$|-1+2i| + |s+1-2i| = -1+2i+s+1-2i = s,$$

completing our proof for this case. □

Proposition 3.21. *For $s \equiv 2 \bmod 4$ and $k = \frac{s^2-8}{2}$, the equation*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0,s]) = 2$$

holds.

Proof. We have already proved that for $s \equiv 2 \bmod 4$, $k = \frac{s^2-16}{2}$ yields a solution to our equation. We now prove that $k = \frac{s^2-8}{2}$ is also a solution.

Our spanning set is $A = \{(0,x),(1,y)\}$, where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$. Because $\lambda_1 \cdot x + \lambda_2 \cdot y \in [\frac{-s^2-4s}{2}, \frac{s^2+4s}{2}]$ for all $(\lambda_1,\lambda_2) \in \mathbb{Z}^2([0,s])$, all elements of $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ are either directly or negatively spanned.

We prove that for all directly spanned $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$, either the element $(a,b+8)$ is also directly spanned or (a,b) can also be negatively spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$, meaning that $(a,b+8)$ will be negatively spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-8}$.

Because every negatively spanned element is the inverse of a directly spanned element, by symmetry the above suffices to prove, for all negatively spanned $(a,b) \in$

$\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$, that $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-8}$ is spanned, while $(a, b+8) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-8}$ will be spanned by the (λ_1, λ_2) that negatively spanned $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$.

Given some $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ that directly spans some $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$, we will prove the above claim about $(a, b+8) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-8}$ by applying one of the following formulae to λ_1, λ_2

1. The coefficients $\mu_1 = \lambda_1 - 2,\ \mu_2 = \lambda_2 + 2$ will directly span the element $(a, b+8)$;
2. The coefficients $\mu_1 = \lambda_1,\ \mu_2 = \lambda_2 - 4x = \lambda_2 - (2s+8)$ negatively span the element $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ and therefore negatively span $(a, b+8) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-8}$;
3. The coefficients $\mu_1 = \lambda_1 + s + 2,\ \mu_2 = \lambda_2 + 6 - s$ directly span the element $(a, b+8)$.

While all of the above three coefficient pairs span the given elements, there is no guarantee that $(\mu_1, \mu_2) \in \mathbb{Z}^2([0, s])$. We now prove that in all cases, at least one of these pairs is within the bounds, keeping in mind that we are presuming the initial coefficients *directly* span the element (a, b).

The first rule will work in many cases. If $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ is directly spanned by λ_1, λ_2 such that $|\lambda_1| + |\lambda_2| \leq s - 4$, then

$$|\mu_1| + |\mu_2| \leq |\lambda_1| + |\lambda_2| + 4 \leq s - 4 + 4 = s,$$

and the coefficients are within bounds. Further, for any $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ where either $\lambda_1 \geq 2$ or $\lambda_2 \leq -2$, it is guaranteed that

$$|\mu_1| + |\mu_2| \leq |\lambda_1| + |\lambda_2| + 2 - 2 \leq s.$$

The second formula is guaranteed to stay in bounds if $\lambda_2 \geq s - 4$ because this implies that $|\lambda_2 - 2s + 8| \leq |-s + 4| \leq |\lambda_2|$, and consequently

$$|\mu_1| + |\mu_2| \leq |\lambda_1| + |\lambda_2| \leq s.$$

If $\lambda_1 \in \{-1, 0, 1\}$, then the first formula works when $\lambda_2 \leq s - 5$ and the second formula works when $\lambda_2 \geq s - 4$.

If $\lambda_2 = -1$, then $\lambda_1 \geq 2$ because we assumed the coefficients directly spanned an element, so formula 1 works. For similar reasons formula 1 also works whenever $\lambda_2 = 0$, so we assume below that $\lambda_2 \geq 1$.

Keeping track of what we have already proved, we may now assume that the coefficients directly spanning $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ are such that $\lambda_1 \leq -2$, $\lambda_2 \in [1, s-5]$, and $|\lambda_1| + |\lambda_2| \geq s-3$. We first address the particular cases where $\lambda_2 = s-5$ before continuing.

We know that $\lambda_1 \leq -2$, so when $\lambda_2 \in \{-3, -2\}$ formula 2 works. On the other hand, when $\lambda_1 \in \{-5, -4\}$, we can apply formula 3 and the coefficients stay in the bounds.

We may now safely assume further that $\lambda_2 \in [1, s-6]$, which along with the fact that $\lambda_1 \leq -2$ and $|\lambda_1| + |\lambda_2| = \lambda_2 - \lambda_1 \geq s-3$ implies

$$\begin{aligned} \lambda_2 - \lambda_1 &\geq s - 3 \\ -s &\geq \lambda_1 - \lambda_2 - 3 > \lambda_1 - \lambda_2 - 4 \\ s &\geq \lambda_1 + s + 2 - \lambda_2 + s - 6 \\ s &\geq |\lambda_1 + s + 2| + |\lambda_2 + 6 - s|. \end{aligned}$$

Therefore formula 3 yields coefficients $(\mu_1, \mu_2) \in \mathbb{Z}^2([0, s])$.

We have proven our claim about directly spanned elements, which we established above suffices to prove our claim. □

Our work above and Park's result on even s and $k = \frac{s^2}{2}$ in [2] prove Theorem 2.2.

4 Future work

It remains to prove or disprove Conjecture 2.3. We also pose the following general question:

For a given positive integer s, what is the largest group G of rank two such that $\phi_\pm(G, [0, s]) = 2$?

We already know that it is not always a group of the form $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$. For example in the case of $s = 4$ the largest group is $G = \mathbb{Z}_3 \times \mathbb{Z}_{12}$, while the largest spanned group of our form is $\mathbb{Z}_2 \times \mathbb{Z}_{16}$.

Acknowledgments. I would like to thank Haesoo Park for his previous work on the topic, and Professor Béla Bajnok for suggesting I pick up where he left off.

References

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[2] Haesoo Park. "Finding a Pair that Spans $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$". In: *Research Papers in Mathematics* 22 (2020). Ed. by Béla Bajnok.