$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$ Draft

William Kyle Beatty

Gettysburg College
Gettysburg, PA 17325-1486 USA
E-mail: research@wkylebeatty.com

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1 Definitions and Previous Results

Definition 1. *For a positive m and a nonnegative h, a* layer *of the m-dimensional integer lattice is defined as*

$$\mathbb{Z}^m(h) = \{(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}^m : |\lambda_1| + |\lambda_2| + \dots + |\lambda_m| = h\}.$$

For a given $s \geq 0$, we also employ an interval notation to describe subsets of the integer lattice

$$\mathbb{Z}^m([0, s]) = \{(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}^m : |\lambda_1| + |\lambda_2| + \dots + |\lambda_m| \in [0, s]\}.$$

Definition 2. *Let s be a positive integer and let $A = \{a_1, a_2, \dots, a_m\}$. The $[0, s]$-fold signed sumset of A is defined as*

$$[0, s]_{\pm}A = \{\lambda_1 \cdot a_1 + \lambda_2 \cdot a_2 + \cdots + \lambda_m \cdot a_m : (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}^m([0, s])\}.$$

Definition 3. *Let s be a positive integer, G be a group, and A a subset of G. Then A* spans *G if and only if $[0, s]_{\pm}A = G$. In this case we call A a* spanning set *of G, and denote by $\phi_{\pm}$ the size of the smallest spanning set of G for a given s:*

$$\phi_{\pm}(G, [0, s]) = \min\{|A| : [0, s]_{\pm}A = G\}.$$

Our work focuses on groups of the form $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ for which $\phi_{\pm}(G, [0, s]) = 2$. We include here Park's results in [2]:

Theorem 4 (Park, 2020)**.** *Given a positive integer s, let $k = \frac{s^2}{2}$ when s is even and $k = \frac{s^2-1}{2}$ when s is odd. Then $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$, where the spanning set of $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ is $\{(0, 1), (1, s-1)\}$ when s is even and $\{(1, \frac{s-1}{2}), (1, \frac{s+1}{2})\}$ when s is odd.*

Conjecture 5 (Park, 2020)**.** *The value of k found in the theorem above is the largest possible k for which $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$.*

2 Summary of New Findings

We proved the above conjecture in [1], placing a sharp upper bound on solutions k to the equation for a given s.

Our main results are propositions 11, 13, 15, 19, and 20, all of which are made possible by theorem 6 below. Together they demonstrate the following solutions:

- s is odd: $\{k \in \mathbb{N} : k \leq \frac{s^2-1}{2}\}$
- $s \equiv 0 \bmod 4$: $\{k \in \mathbb{N} : k \leq \frac{s^2-s}{2}\} \cup \{k \in \mathbb{N} : k \leq \frac{s^2}{2}$ and k is even$\}$
- $s \equiv 2 \bmod 4$: $\{k \in \mathbb{N} : k \leq \frac{s^2-s}{2}\} \cup \{k \in \mathbb{N} : k \leq \frac{s^2-16}{2}$ and $k \equiv 2 \bmod 4\} \cup \{k = \frac{s^2}{2}\}$.

We are in the process of proving the case where $s \equiv 2 \bmod 4$ and $k = \frac{s^2-8}{2}$, which we conjecture to be the final class of solutions.

Because of Park's conjectured upper bound which we proved in [1], the case for odd s is solved comprehensively. There is only a linear amount of cases to be proved (either as solutions or non-solutions) for even s compared to the quadratic number of possible solutions.

3 Results

Theorem 6. *Let s be a positive integer, and let d, x, y be positive integers such that*

- *$s^2 - d^2$ is even*
- *x is odd*
- *$x + y = s$*
- *x and y are coprime*
- *$4xy = s^2 - d^2$*

then the group $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$ is s-spanned by the pair of elements $\{(0,x),(1,y)\}$; therefore

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}, [0,s]) = 2.$$

Proof. For an arbitrary element $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$, we first show that there are coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = (a,b)$.

The span of $(0,x)$ will form a subgroup $H \leq G$ of order $\frac{s^2-d^2}{x} = 4y$. This subgroup has $\frac{|G|}{4y} = 2x$ corresponding cosets. The element (a,b) that we wish to span must lie in one of these cosets, so we first show that each of the cosets can be reached by some multiple $\lambda_2 \cdot (1,y)$.

For each $\mu \in [0, 2x-1]$, the multiple $\mu \cdot (1,y)$ reaches a different coset of H, implying that this set of multiples reaches all $2x$ cosets of H: take two distinct $\mu_1, \mu_2 \in [0, 2x-1]$ and assume without loss of generality that $\mu_1 > \mu_2$. $\mu_1 \cdot (1,y)$ and $\mu_2 \cdot (1,y)$ belong to different cosets because $\mu_1 \cdot (1,y) - \mu_2 \cdot (1,y) = (\mu_1 - \mu_2) \cdot (1,y) \notin H$. To see this, let $\mu' = \mu_1 - \mu_2 \in [1, 2x-1]$ and suppose for contradiction that $\mu' \cdot (1,y) \in H$. This would imply that

$$\mu' \cdot (1,y) = c \cdot (0,x)$$

for some integer c. Because x and y are coprime, the only $\mu' \in [1, 2x-1]$ that could satisfy the above equation is x. But because x is odd, we know that

$$x \cdot (1,y) = (1,xy) \neq c \cdot (0,x)$$

for any c. We therefore conclude that $\mu \cdot (1,y)$ spans a different coset of H for each $\mu \in [0, 2x-1]$, and consequently that they span every coset.

We return to our element $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$. It must lie in some coset of H, so by our findings above there must be some $\lambda_2 \in [0, 2x-1]$ such that $\lambda_2 \cdot (1,y)$ is in this same coset. Because each of these cosets is of size $4y$, there must be some $\lambda_1 \in [-2y+1, 2y]$ such that

$$\lambda_1 \cdot (0, x) + \lambda_2 \cdot (1, y) = (a, b).$$

There is no guarantee, however, that $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$. Based on the constraints above, we have only that

$$|\lambda_1| + |\lambda_2| \leq 2y + 2x - 1 = 2s - 1.$$

If $|\lambda_1| + |\lambda_2| \leq s$, then we have found coefficients in $\mathbb{Z}^2([0, s])$ that span (a, b) and are done.

If, however, $|\lambda_1| + |\lambda_2| \in [s+1, 2s-1]$, we show that there exist $(\lambda_1', \lambda_2') \in \mathbb{Z}^2([0, s])$ that span the same element (a, b). We select these values as follows:

$$\lambda_1' = \begin{cases} \lambda_1 - 2y, & \lambda_1 \geq 0 \\ \lambda_1 + 2y, & \lambda_1 < 0 \end{cases} \qquad \lambda_2' = \lambda_2 - 2x.$$

This selection of variables implies that $|\lambda_1'| = 2y - |\lambda_1|$ and $|\lambda_2'| = 2x - |\lambda_2|$. Therefore

$$\begin{aligned} |\lambda_1'| + |\lambda_2'| &= 2y - |\lambda_1| + 2x - |\lambda_2| \\ |\lambda_1'| + |\lambda_2'| &= 2(x + y) - (|\lambda_1| + |\lambda_2|) \\ |\lambda_1'| + |\lambda_2'| &= 2s - (|\lambda_1| + |\lambda_2|) \,. \end{aligned}$$

Because $|\lambda_1| + |\lambda_2| \in [s+1, 2s-1]$, this implies that

$$|\lambda_1'| + |\lambda_2'| \in [1, s-1],$$

placing (λ_1', λ_2') within the acceptable bounds for $\mathbb{Z}^2([0, s])$.

It remains only to prove that (λ_1', λ_2') span the same element (a, b) as the original coefficients. If $\lambda_1 \geq 0$, meaning $\lambda_1' = \lambda_1 - 2y$, then

$$\begin{aligned} \lambda_1' \cdot (0, x) + \lambda_2' \cdot (1, y) &= (\lambda_1 - 2y) \cdot (0, x) + (\lambda_2 - 2x) \cdot (1, y) \\ &= [\lambda_1 \cdot (0, x) + \lambda_2 \cdot (1, y)] - [2y \cdot (0, x) + 2x \cdot (1, y)] \\ &= (a, b) - (0, 4xy) \\ &= (a, b) - (0, s^2 - d^2) \\ &= (a, b) - (0, 0) \\ \lambda_1' \cdot (0, x) + \lambda_2' \cdot (1, y) &= (a, b). \end{aligned}$$

If $\lambda_1 < 0$, meaning $\lambda_1' = \lambda_1 + 2y$, then

$$\begin{aligned}\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (\lambda_1 + 2y) \cdot (0,x) + (\lambda_2 - 2x) \cdot (1,y) \\ &= [\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y)] - 2y \cdot (0,x) + 2x \cdot (1,y) \\ &= (a,b) - (0,2xy) + (0,2xy) \\ \lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (a,b).\end{aligned}$$

Since in either case, the new $(\lambda_1', \lambda_2') \in \mathbb{Z}^2([0,s])$ spans the same element (a,b), we have that our arbitrary element $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-d^2}$ is s-spanned by the elements $(0,x)$ and $(1,y)$, as was to be shown. □

Lemma 7. *Let s be a positive integer, $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ be a group, and $A = \{p,q\}$ be a pair of elements. Then A is a $[0,s]$ signed spanning set for G if and only if it spans the subset $\mathbb{Z}_2 \times \{0,1,\ldots,k\} \subset G$.*

Proof. The "only if" direction is clearly true, so we prove the "if" statement. For any $g \in G$, either g or $-g$ is in the set $\mathbb{Z}_2 \times \{0,1,\ldots,k\}$ In the latter case, take the coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$ that span $-g$ and observe that

$$\begin{aligned}-\lambda_1 \cdot p + -\lambda_2 \cdot q &= -(\lambda_1 \cdot p + \lambda_2 \cdot q) \\ -\lambda_1 \cdot p + -\lambda_2 \cdot q &= -(-g) \\ -\lambda_1 \cdot p + -\lambda_2 \cdot q &= g.\end{aligned}$$

Therefore g can also be spanned by the spanning set A, proving our claim. □

Definition 8. *Given some $s \geq 1$, a group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$, a pair of elements $A = \{(a,x),(b,y)\}$, and some $(c,z) \in G$, we say that the element (c,z) is **directly spanned** if there exist $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$ such that*

$$\lambda_1 \cdot a + \lambda_2 \cdot b \equiv c \bmod 2 \quad \textit{and } \lambda_1 \cdot x + \lambda_2 \cdot y = z.$$

Definition 9. *Given some $s \geq 1$, a group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$, a pair of elements $A = \{(a,x),(b,y)\}$, and some $(c,z) \in G$, we say that the element (c,z) is **negatively spanned** if there exist $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0,s])$ such that*

$$\lambda_1 \cdot a + \lambda_2 \cdot b \equiv c \bmod 2 \quad \textit{and } \lambda_1 \cdot x + \lambda_2 \cdot y = -1 \cdot 2k + z.$$

Lemma 10. *Let $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ be a group with a pair of elements $A = \{(a,x),(b,y)\}$ that directly s-span some element $(c,z) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. Then for any $k' \in \mathbb{N}$ the corresponding element $(c,z) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k'}$ is also directly s-spanned by A.*

Proof. The first component of the element spanned $\lambda_1 \cdot a + \lambda_2 \cdot b \equiv c \bmod 2$ will clearly not change between the two groups, and by the definition of direct spanning we have that

$$\lambda_1 \cdot x + \lambda_2 \cdot y = z \equiv z \bmod 2k',$$

so the element (c, z) is directly spanned in both groups. □

Proposition 11. *Given an odd s, the equation*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$$

holds if and only if $k \in [1, \frac{s^2-1}{2}]$.

Proof. Given some odd s, we let $k = \frac{s^2-1}{2}$ and let

$$x = \begin{cases} \frac{s+1}{2}, & s \equiv 1 \bmod 4 \\ \frac{s-1}{2}, & s \equiv 3 \bmod 4 \end{cases} \qquad y = \begin{cases} \frac{s-1}{2}, & s \equiv 1 \bmod 4 \\ \frac{s+1}{2}, & s \equiv 3 \bmod 4. \end{cases}$$

Our choice of x and y satisfies the hypothesis of theorem 6, which we apply to prove that the set $A = \{(0, x), (1, y)\}$ s-spans $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-1}$. We have already proven in [1] that any solution with odd s must have $k \leq \frac{s^2-1}{2}$, so we next prove that the subset $\mathbb{Z}_2 \times \{0, 1, \ldots, \frac{s^2-1}{2} - 1\}$ is directly spanned by A, which by lemmas 7 and 10 suffices to prove our claim.

For a given $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ that spans a certain element of $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-1}$, we let μ_1 be the coefficient corresponding to $\frac{s-1}{2}$ and μ_2 be the one corresponding to $\frac{s+1}{2}$, i.e.

$$\mu_1 = \begin{cases} \lambda_2, & s \equiv 1 \bmod 4 \\ \lambda_1, & s \equiv 3 \bmod 4 \end{cases} \qquad \mu_2 = \begin{cases} \lambda_1, & s \equiv 1 \bmod 4 \\ \lambda_2, & s \equiv 3 \bmod 4. \end{cases}$$

Because $\mu_1 \cdot \frac{s-1}{2} + \mu_2 \cdot \frac{s+1}{2} \geq \frac{-s^2-s}{2}$ for all $(\mu_1, \mu_2) \in \mathbb{Z}^2([0, s])$, then for any $(a, b) \in \mathbb{Z}_2 \times \{0, 1, \ldots, \frac{s^2-1}{2} - 1\}$ that is negatively spanned by such a (μ_1, μ_2), we have that $b \in [\frac{s^2-s-2}{2}, \frac{s^2-1}{2} - 1]$. For a negatively spanned b in this range we know that $\mu_1 + \mu_2 = -s$. For suppose that $\mu_1 + \mu_2 \geq -s + 1$, and observe that

$$\mu_1 \cdot \frac{s-1}{2} + \mu_2 \cdot \frac{s+1}{2} \geq (-s+1) \cdot \frac{s+1}{2} = \frac{-s^2+1}{2} \equiv \frac{s^2-1}{2} \bmod s^2 - 1,$$

which is outside of our established range for negatively spanned b.

For a given (a,b) negatively spanned by some $(\mu_1,\mu_2) \in \mathbb{Z}^2([0,s])$, we divide the remaining work into two cases. In the case where $\mu_2 = -s$ and therefore $\mu_1 = 0$, we have that

$$(s-2) \cdot \frac{s+1}{2} = \frac{s^2-s-2}{2}$$

which is equivalent $\bmod s^2 - 1$ to

$$-s \cdot \frac{s+1}{2} = \frac{-s^2-s}{2} \equiv \frac{s^2-s-2}{2} \bmod s^2 - 1.$$

Furthermore, because $-s \equiv (s-2) \bmod 2$ the coefficients $\mu_1' = 0, \mu_2' = s-2$ will directly span the same element of $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-1}$ which the coefficients $\mu_1 = 0, \mu_2 = -s$ negatively span.

In the second case, where $\mu_2 \geq -s+1$ and therefore $\mu_1 \leq -1$, let $\mu_1' = \mu_1 + s + 1$ and $\mu_2' = \mu_2 + s - 1$. We first note that

$$\begin{aligned}
\mu_1' \cdot \frac{s-1}{2} + \mu_2' \cdot \frac{s+1}{2} &= (\mu_1 + s + 1) \cdot \frac{s-1}{2} + (\mu_2 + s - 1) \cdot \frac{s+1}{2} \\
&= \left(\mu_1 \cdot \frac{s-1}{2} + \mu_2 \cdot \frac{s+1}{2}\right) + \frac{(s+1)(s-1)}{2} + \frac{(s-1)(s+1)}{2} \\
&= b - (s^2-1) + (s^2-1) \\
&= b.
\end{aligned}$$

Taken together with the fact that $\mu_1' \equiv \mu_1 \bmod 2$ and $\mu_2' \equiv \mu_2 \bmod 2$, the above implies that (μ_1', μ_2') directly spans the element (a,b) in question. We now prove that $(\mu_1', \mu_2') \in \mathbb{Z}^2([0,s])$, keeping in mind that $\mu_2 \geq -s+1$

$$\begin{aligned}
|\mu_1'| + |\mu_2'| &= |\mu_1 + s + 1| + |\mu_2 + s - 1| \\
&= (\mu_1 + s + 1) + (\mu_2 + s - 1) \\
&= (\mu_1 + \mu_2) + s + 1 + s - 1 \\
&= -s + 2s - 2 \\
&= s - 2.
\end{aligned}$$

We have shown that our new coefficients $(\mu_1', \mu_2') \in \mathbb{Z}^2([0,s])$ directly span the element in question (a,b). This proves that any element in our subset is directly spanned by A, which as shown above suffices to prove our claim. □

Proposition 12. *For a given positive integer* $s \equiv 0 \bmod 4$*, let* $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$*. Then the pair of elements* $A = \{(0,x),(1,y)\}$ *where*

$$x = \frac{s-2}{2} \quad \text{and} \quad y = \frac{s+2}{2}$$

is an s-spanning pair for G. Therefore

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}, [0, s]) = 2.$$

Proof. Our proposition is a particular case of theorem 6, where $d = 2$. Because $s \equiv 0 \bmod 4$, we know that x and y are both odd; because they differ by 2, this further implies that they are coprime. The hypothesis of theorem 6 thus holds, proving our claim. □

Proposition 13. *For a given positive integer $s \equiv 0 \bmod 4$, take any $k \leq \frac{s^2-s}{2}$. The group $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ can be s-spanned by the pair $A = \{(0, x), (1, y)\}$ where*

$$x = \frac{s-2}{2} \quad \text{and} \quad y = \frac{s+2}{2},$$

and consequently $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2$.

Proof. It suffices by lemmas 7 and 10 to show that for $k = \frac{s^2-s}{2}$, the subset $\mathbb{Z}_2 \times \{0, 1, \ldots, k\}$ can be *directly spanned* by A. We proved above in proposition 12 that the group $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ is spanned by A. Observe that for any $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$

$$\frac{-s^2 - 2s}{2} \leq \lambda_1 \cdot x + \lambda_2 \cdot y \leq \frac{s^2 + 2s}{2},$$

so any element $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ that is not directly spanned will instead be negatively spanned by definition 9 above. Taking an arbitrary $(a, b) \in \mathbb{Z}_2 \times \{0, \ldots, \frac{s^2-s}{2}\}$ that is negatively spanned by some $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$, we will show that this same element is directly spanned by the coefficients $(\lambda_1, \lambda_2 + 4x) \in \mathbb{Z}^2([0, s])$. We first observe that

$$\begin{aligned}
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= (\lambda_1 \cdot x + \lambda_2 \cdot y) + 4x \cdot y \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= -1 \cdot (s^2 - 4) + b + s^2 - 4 \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= b.
\end{aligned}$$

Because $4x$ is even, we also have that $\lambda_2 + 4x \equiv a \bmod 2$, meaning the new coefficients directly span (a, b). It still remains to be shown that the new coefficients are in $\mathbb{Z}^2([0, s])$.

Because the element is negatively spanned and $b \in [0, \frac{s^2-s}{2}]$, we know that its coefficients were generated by the second step in theorem 6, so $\lambda_2 < 0$ and $|\lambda_1| +$

$|\lambda_2| \in [1, s-1]$. We will prove that $\lambda_2 \leq -2x$, which implies that the above coefficients are also in the bounds i.e.

$$|\lambda_1| + |\lambda_2 + 4x| \leq s.$$

Suppose that $\lambda_2 = -2x + 1 = -s + 3$ and that $\lambda_1 = -2$. This is the coefficient pair with the lowest spanned value $\lambda_1 \cdot x + \lambda_2 \cdot y$ such that $-2x < \lambda_2 < 0$ and $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([1, s-1])$. Calculating this value

$$\begin{aligned}
\lambda_1 \cdot x + \lambda_2 \cdot y &= -2 \cdot \frac{s-2}{2} + (-s+3) \cdot \frac{s+2}{2} \\
&= -s + 2 + \frac{-s^2 + s + 6}{2} \\
&= \frac{-s^2 - s + 10}{2} \\
&\equiv \frac{s^2 - s + 2}{2} \bmod s^2 - 4,
\end{aligned}$$

we see that it is outside the assumed range $b \in [0, \frac{s^2-s}{2}]$. Therefore for all $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([1, s-1])$ that negative span an element with b in this range, we know that $\lambda_2 \leq -2x$. Hence $(\lambda_1, \lambda_2 + 4x) \in \mathbb{Z}^2([0, s])$ directly spans the element (a, b) while staying within the bounds for spanning coefficients, which as shown above suffices to prove our claim. □

Lemma 14. *Let $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ such that $b \in [\frac{s^2-s+2}{2}, \frac{s^2+s-4}{2}]$. If (a, b) is negatively spanned by the pair A that spans $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$, then A also negatively spans $(a, b+4)$.*

Proof. Take any negatively spanned element (a, b) within the specified range, and consider its spanning coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$. We first observe that the coefficients $(\lambda_1 - 2, \lambda_2 + 2)$ will span the element $(a, b+4)$, as $\lambda_2 + 2 \equiv \lambda_2 \equiv a \bmod 2$, and

$$\begin{aligned}
(\lambda_1 - 2) \cdot \frac{s-2}{2} + (\lambda_1 + 2) \cdot \frac{s+2}{2} &= b - 2 \cdot \frac{s-2}{2} + 2 \cdot \frac{s+2}{2} \\
(\lambda_1 - 2) \cdot \frac{s-2}{2} + (\lambda_1 + 2) \cdot \frac{s+2}{2} &= b + 4.
\end{aligned}$$

We now prove that these coefficients are also in $\mathbb{Z}^2([0, s])$. We begin by proving that $\lambda_2 \leq -2$. First, if $\lambda_2 = 0$, then any value of λ_1 can not span (a, b), for

$$\begin{aligned}\lambda_1 \cdot x + 0 \cdot y &\geq -s \cdot \frac{s-2}{2} \\ &= \frac{-s^2+2s}{2} \\ &\equiv \frac{s^2+2s-8}{2} \bmod s^2-4\end{aligned}$$

which is outside of the specified range for b. Second, if $\lambda_2 \neq 0$ but $\lambda_2 \geq -1$, then $\lambda_1 \geq -s+1$ which implies

$$\begin{aligned}\lambda_1 \cdot x + \lambda_2 \cdot y &\geq (-s+1) \cdot \frac{s-2}{2} + -1 \cdot \frac{s+2}{2} \\ &= \frac{-s^2+3s-2}{2} - \frac{s+2}{2} \\ &= \frac{-s^2+2s-4}{2} \\ &\equiv \frac{s^2+2s-12}{2} \bmod s^2-4,\end{aligned}$$

which is also outside of the specified range for b. We have proved that $\lambda_2 \leq -2$, implying that $|\lambda_2 + 2| = |\lambda_2| - 2$. Clearly we also have that $|\lambda_1 - 2| \leq |\lambda_1| + 2$, meaning

$$|\lambda_1 - 2| + |\lambda_2 + 2| \leq |\lambda_1| + 2 + |\lambda_2| - 2 = |\lambda_1| + |\lambda_2| \leq s.$$

Therefore $(\lambda_1 - 2, \lambda_2 + 2) \in \mathbb{Z}^2([0, s])$, proving our claim. □

Proposition 15. *For a positive integer* $s \equiv 0 \bmod 4$*, let* k *be an even integer* $k \leq \frac{s^2-4}{2}$*. Then the pair* A *from above* s*-spans* $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ *implying*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2.$$

Proof. Take some $i \in \mathbb{N}$, and let $k_i = \frac{s^2-4-4i}{2}$. We know by proposition 13 that all elements (a, b) with $b \leq \frac{s^2-s}{2}$ can be directly spanned by A, and thus are spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ for any value of k. Next, we prove that A spans any $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$ such that $b \in [\frac{s^2-s+2}{2}, k_i]$. We first note that if such an element exists, then $4i < s - 4$. If this element is directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$, then it is also directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. If it is negatively spanned, then a more involved argument is required.

We prove that the coefficients that span $(a, b + 4i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ will span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. First, consider the coefficients (λ_1, λ_2) that negatively span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$.

Because $b \in [\frac{s^2-s+2}{2}, k_i]$, we have that

$$\begin{aligned} k_i &\geq \frac{s^2-s+2}{2} \\ \frac{s^2-4-4i}{2} &\geq \frac{s^2-s+2}{2} \\ 4i &< s-4. \end{aligned}$$

We inductively apply lemma 14 up to $(a, b+4i)$ and call its spanning coefficients (μ_1, μ_2). The lemma holds for all $b+4,\ b+8,\ \dots, b+4i$ because $4i < s-4$ implies that $b + 4i < \frac{s^2+s-6}{2}$, within the range where lemma 14 applies.

Finally, we show that the coefficients (μ_1, μ_2) that span $(a, b+4i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$ will span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. Clearly the value a of the spanned element will not change between the two groups, as the spanning pair A and the value μ_2 have not. Next, because the coefficients negatively span $(a, b+4i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-4}$, we have

$$\begin{aligned} \mu_1 \cdot \frac{s-2}{2} + \mu_2 \cdot \frac{s+2}{2} &= -1 \cdot (s^2-4) + b + 4i \\ \mu_1 \cdot \frac{s-2}{2} + \mu_2 \cdot \frac{s+2}{2} &= -1 \cdot (s^2-4-4i) + b, \end{aligned}$$

meaning $(\mu_1, \mu_2) \in \mathbb{Z}^2([0, s])$ will span the element $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. □

Proposition 16. *Let $s \equiv 2 \bmod 4$ be a positive integer, and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$. Then the pair of elements $A = \{(0, x), (1, y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$ spans G, meaning*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}, [0, s]) = 2.$$

Proof. Our proposition is a particular case of theorem 6, where $d = 4$. Because $s \equiv 2 \bmod 4$, we know that x and y are both odd; because they differ by 4, this further implies that they are coprime. The hypothesis of theorem 6 thus holds, proving our claim. □

Proposition 17. *Let $s \equiv 2 \bmod 4$ be a positive integer, and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$. Then any $(a, b) \in G$ with $b \in \{0, 1, \dots, \frac{s^2-4s+6}{2}\}$ can be directly spanned by $A = \{(0, x), (1, y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$.*

Proof. Proposition 16 establishes that A spans the group G. Each element in our specified subset is either directly or negatively spanned, so we show that the negatively spanned ones have another set of spanning coefficients in $\mathbb{Z}^2([0, s])$ that directly span them.

Let $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ be the coefficients that negatively span some element (a, b) with $b \leq \frac{s^2-4s+6}{2}$. We first show that $\lambda_2 \leq -2x$. Assume for contradiction that $\lambda_2 \geq -2x + 1 = -s + 5$, and therefore that $\lambda_1 \geq -5$. This implies that

$$\begin{aligned}
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq (-5) \cdot \frac{s-4}{2} + (-s+5) \cdot \frac{s+4}{2} \\
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq \frac{-5s+20}{2} - \frac{s^2-s-20}{2} \\
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq \frac{-s^2-4s+40}{2} \equiv \frac{s^2-4s+8}{2} \bmod s^2-16,
\end{aligned}$$

which is higher than the assumed value $b \leq \frac{s^2-4s+6}{2}$. Therefore $\lambda_2 \leq -2x$, meaning $|\lambda_1| + |\lambda_2 + 4x| \leq |\lambda_1| + |\lambda_2| \leq s$.

We have established that $(\lambda_1, \lambda_2 + 4x) \in \mathbb{Z}^2([0, s])$, and now show that these new coefficients directly span (a, b). Because λ_2 and $\lambda_2 + 4x$ have the same parity, the first component a of the spanned element remains unchanged. To see that the same b is directly spanned, observe that

$$\begin{aligned}
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= (\lambda_1 \cdot x + \lambda_2 \cdot y) + 4x \cdot y \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= -1 \cdot (s^2 - 16) + b + s^2 - 16 \\
\lambda_1 \cdot x + (\lambda_2 + 4x) \cdot y &= b.
\end{aligned}$$

Therefore the same element (a, b) is also directly spanned by the pair A, as was to be shown. □

Lemma 18. *For some $s \equiv 2 \bmod 4$, let $G = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ and $A = \{(0, x), (1, y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$. For any $(a, b) \in G$ with $b \in [\frac{s^2-4s+8}{2}, \frac{s^2+4s-42}{2}]$ that is negatively spanned by coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$, the coefficients $(\lambda_1 - 2, \lambda_2 + 2)$ negatively span the element $(a, b + 8)$ and are also in $\mathbb{Z}^2([0, s])$.*

Proof. We first show that if the coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$ negatively span (a, b), then $\lambda_2 \leq -2$. Supposing for contradiction that it isn't, we split the scenario into two cases: $\lambda_2 = 0$ and $\lambda_2 \geq -1$ but $\lambda_2 \neq 0$.

In the first case, we have that $\lambda_1 \geq -s$, implying

$$\begin{aligned}
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq -s \cdot \frac{s-4}{2} \\
\lambda_1 \cdot x + \lambda_2 \cdot y &\geq \frac{-s^2+4s}{2} \equiv \frac{s^2+4s-32}{2} \bmod s^2-16,
\end{aligned}$$

which exceeds our presumed range for b. In the second case where $\lambda_2 \neq 0$, we must have $\lambda_1 \geq -s+1$ and therefore

$$\lambda_1 \cdot x + \lambda_2 \cdot y \geq -s+1 \cdot \frac{s-4}{2} + -1 \cdot \frac{s+4}{2}$$
$$\lambda_1 \cdot x + \lambda_2 \cdot y \geq \frac{-s^2+5s-4}{2} + \frac{-s-4}{2}$$
$$\lambda_1 \cdot x + \lambda_2 \cdot y \geq \frac{-s^2+4s-8}{2} \equiv \frac{s^2+4s-40}{2} \bmod s^2-16,$$

which also exceeds our presumed range for b. Therefore we must have $\lambda_2 \leq -2$. This bound implies that

$$|\lambda_1 - 2| + |\lambda_2 + 2| \leq |\lambda_1| + 2 + |\lambda_2| - 2 = |\lambda_1| + |\lambda_2| \leq s$$

and thus $(\lambda_1 - 2, \lambda_2 + 2) \in \mathbb{Z}^2([0, s])$. To conclude our argument we show that these coefficients span $(a, b+8)$. First, $\lambda_2 + 2 \equiv a \bmod 2$ because it has the same parity as λ_2, and thus the first component a is still spanned. For the second component b we have

$$\begin{aligned}(\lambda_1 - 2) \cdot x + (\lambda_2 + 2) \cdot y &= (\lambda_1 \cdot x + \lambda_2 \cdot y) - 2 \cdot x + 2 \cdot y \\ &= b - (s^2 - 16) - (s-2) + (s+2) \\ &= b - (s^2 - 16) + 4 \\ &= b + 4 - (s^2 - 16) \equiv b + 4 \bmod s^2 - 16.\end{aligned}$$

□

Proposition 19. *Take any $i \geq 0$, and let $k_i = \frac{s^2-16}{2} - 4i$. Then the group $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$ is s-spanned by $A = \{(0, x), (1, y)\}$ where $x = \frac{s-4}{2}$ and $y = \frac{s+4}{2}$.*

Proof. Proposition 16 proves the case where $i = 0$. Now consider some $i \geq 1$ and its corresponding k_i and $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. We will prove that the subset $\mathbb{Z}_2 \times \{0, \ldots, k_i\}$ of this group is spanned by A, which suffices by lemma 7 to prove that A spans the entire group. Any elements (a, b) of this subset that can be directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ will also be directly spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$, so we focus our attention on elements that can only be negatively spanned in our subset of $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$. By the above proposition, this implies that $b \in [\frac{s^2-4s+8}{2}, k_i]$. If such a b exists, it follows that

$$\begin{aligned}\frac{s^2-4s+8}{2} &\leq k_i = \frac{s^2-16-8i}{2} \\ s^2 - 4s + 8 &\leq s^2 - 16 - 8i \\ 8i &\leq 4s - 24.\end{aligned}$$

We show by induction that $(a, b + 8i) \in \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ can also be negatively spanned. We repeatedly apply lemma 18 to (a, b), then $(a, b + 8)$, and so on up to $(a, b+8(i-1))$ to prove that $(a, b+8i)$ is negatively spanned. The hypothesis of the lemma holds for all relevant values $b + 8, \dots, b + 8(i - 1)$ because by our inequality above and the given range for b we have

$$\begin{aligned} b + 8(i-1) &\leq k_i + 8(i-1) \\ b + 8(i-1) &\leq \frac{s^2 - 16 - 8i}{2} + 8(i-1) \\ b + 8(i-1) &\leq \frac{s^2 + 8i - 32}{2} \\ b + 8(i-1) &\leq \frac{s^2 + 4s - 56}{2} < \frac{s^2 + 4s - 40}{2}, \end{aligned}$$

placing $b + 8(i - 1)$ within the acceptable range.

We have proven that $(a, b+8i)$ can be negatively spanned in $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-16}$ by some coefficients $(\lambda_1, \lambda_2) \in \mathbb{Z}^2([0, s])$. These coefficients will also span $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i} = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-16-8i}$, as they will clearly span the same value a for the first component, and for the second component will span

$$\begin{aligned} \lambda_1 \cdot x + \lambda_2 \cdot y &= -1 \cdot (s^2 - 16) + b + 8i \\ \lambda_1 \cdot x + \lambda_2 \cdot y &= -1 \cdot (s^2 - 16 - 8i) + b, \end{aligned}$$

consequently spanning $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$. Therefore every element in $\mathbb{Z}_2 \times \{0, \dots, k_i\}$ is either directly spanned or negatively spanned by our pair A, implying that $\mathbb{Z}_2 \times \mathbb{Z}_{2k_i}$ is s-spanned by A. $\square$

Proposition 20. *Given some $s \equiv 2 \bmod 4$ and any $k \leq \frac{s^2-s}{2}$, let $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. Then the subset $A = \{(0, x), (1, y)\}$ where $x = \frac{s}{2}$ and $y = \frac{s-2}{2}$ s-spans G, meaning*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k}, [0, s]) = 2.$$

Proof. Because $s - 1$ is odd, we apply proposition 11 and find that the set A $(s-1)$-spans the group $\mathbb{Z}_2 \times \mathbb{Z}_{s^2-2s}$. We will prove that A directly s-spans the subset of this group $\mathbb{Z}_2 \times \{0, 1, \dots, \frac{s^2-s}{2}\}$, which suffices to prove our claim by lemmas 7 and 10.

By proposition 11, the subset $\mathbb{Z}_2 \times \{0, 1, \dots, \frac{s^2-2s}{2}\}$ is already known to be directly spanned. We divide the rest of the elements in our subset of interest into four categories:

1. $(0, \frac{s^2-2s}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s-2}{4}$;
2. $(1, \frac{s^2-2s}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s-2}{4}$;
3. $(0, \frac{s^2-2s-2}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s+2}{4}$; and
4. $(1, \frac{s^2-2s-2}{2} + 2i)$ for $i \in \mathbb{N}$ such that $i \leq \frac{s+2}{4}$.

We prove that each subset in turn can be directly s-spanned by A.

Case 1

First, note that for a given $i \leq \frac{s-2}{4}$, we have that

$$\begin{aligned} 2i \cdot x + (s-2i) \cdot y &= \frac{2i \cdot s}{2} + \frac{(s-2i)(s-2)}{2} \\ &= \frac{2i \cdot s}{2} + \frac{s^2 - 2s - 2is + 4i}{2} \\ &= \frac{s^2 - 2s}{2} + 2i, \end{aligned}$$

and that $s + 2i \equiv 0 \bmod 2$. Therefore the coefficients $\lambda_1 = 2i,\ \lambda_2 = s - 2i$ span the desired element. To see that they are in $\mathbb{Z}^2([0, s])$, observe that for $i \leq \frac{s-2}{4}$ we have

$$|2i| + |s - 2i| = 2i + s - 2i = s,$$

completing our proof for this case.

Case 2

First note that for a given $i \leq \frac{s-2}{4}$, we have that

$$\begin{aligned} (y+2i) \cdot x + (x-2i) \cdot y &= \frac{(s-2+4i) \cdot s}{4} + \frac{(s-4i) \cdot (s-2)}{4} \\ &= \frac{s^2 - 2s + 4is}{4} + \frac{s^2 - 2s - 4is + 8i}{4} \\ &= \frac{s^2 - 2s}{2} + 2i, \end{aligned}$$

and that $x \equiv 1 \bmod 2$. Therefore the coefficients $\lambda_1 = y,\ \lambda_2 = x$ span the desired element. To see that they are in $\mathbb{Z}^2([0, s])$, observe that for $i \leq \frac{s-2}{4}$ we have

$$|y + 2i| + |x - 2i| = y + 2i + x - 2i = x + y = s - 1,$$

completing our proof for this case.

Case 3

First, note that for a given $i \leq \frac{s+2}{4}$, we have that

$$\begin{aligned}(y-1+2i)\cdot x+(x+1-2i)\cdot y &= (xy-x+2ix)+(xy+y-2iy)\\ &= 2xy+(y-x)+2i(x-y)\\ &= \frac{s^2-2s}{2}-1+2i\end{aligned}$$

and that $x+1-2i \equiv 0 \bmod 2$. Therefore the coefficients $\lambda_1 = y-1+2i,\ \lambda_2 = x+1-2i$ span the desired element. To see that they are in $\mathbb{Z}^2([0,s])$, observe that for $i \leq \frac{s+2}{4}$ we have

$$|y-1+2i|+|x+1-2i| = \frac{s-4+4i}{2}+\frac{s+2-4i}{2} = s-1,$$

completing our proof for this case.

Case 4

First, note that for a given $i \leq \frac{s+2}{4}$, we have that

$$\begin{aligned}(-1+2i)\cdot x+(s+1-2i)\cdot y &= sy+(y-x)+2i(x-y)\\ &= \frac{s^2-2s}{2}-1+2i,\end{aligned}$$

and that $s+1-2i \equiv 1 \bmod 2$. Therefore the coefficients $\lambda_1 = -1+2i,\ \lambda_2 = s+1-2i$ span the desired element. To see that they are in $\mathbb{Z}^2([0,s])$, observe that for $i \leq \frac{s+2}{4}$ we have

$$|-1+2i|+|s+1-2i| = -1+2i+s+1-2i = s,$$

completing our proof for this case. □

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